

ITP-UH-11/94
 hep-th/9409007



September 1994

BRST Charge for the Orthogonal Series of Bershadsky-Knizhnik Quasi-Superconformal Algebras ¹

Sergei V. Ketov ²

*Institut für Theoretische Physik, Universität Hannover
 Appelstraße 2, 30167 Hannover, Germany
 ketov@kastor.itp.uni-hannover.de*

Abstract

The quantum BRST charges for Bershadsky-Knizhnik orthogonal quasi-superconformal algebras are constructed. These two-dimensional superalgebras have the N -extended non-linearly realised supersymmetry and the $SO(N)$ internal symmetry. The BRST charge nilpotency conditions are shown to have a unique solution at $N > 2$, namely, $N = 4$ and $k = -2$, where k is central extension parameter of the Kač-Moody subalgebra. We argue about the existence of a new string theory with the non-linearly realised $N = 4$ world-sheet supersymmetry and negative ‘critical dimension’.

¹Supported in part by the ‘Deutsche Forschungsgemeinschaft’ and the NATO grant CRG 930789

²On leave of absence from: High Current Electronics Institute of the Russian Academy of Sciences, Siberian Branch, Akademichesky 4, Tomsk 634055, Russia

1. Any known critical N -extended fermionic string theory with $N \leq 4$ world-sheet supersymmetries is based on a two-dimensional (2d) linear N -extended superconformal algebra which is gauged [1]. When a number of world-sheet supersymmetries exceeds two, there are more opportunities to build up new string theories, namely, by utilizing 2d non-linear *quasi-superconformal algebras* (QSCAs) which are known to exist for an arbitrary $N > 2$. The QSCAs can be considered on equal footing with the W algebras *without*, however, having currents of spin higher than two. In the past, only one string theory for $N > 2$ was actually constructed by gauging the ‘small’ linear $N = 4$ SCA with $SU(2)$ internal symmetry [1, 2]. Still, it is of interest to know how many different $N = 4$ string theories exist at all. Any $N = 4$ string constraints are going to be very strong, so that their explicit realisation should always imply non-trivial interplay between geometry, conformal invariance and extended supersymmetry. The $N = 4$ strings are also going to be relevant in the search for the ‘universal string theory’ [3]. In addition, strings with $N = 4$ supersymmetry are expected to have deep connections with integrable models [4, 5], so that we believe they are worthy to be studied.

The full classification of QSCAs is known due to by Fradkin and Linetsky [6]. In particular, the $osp(N|2; \mathbf{R})$ and $su(1,1|N)$ series of QSCAs with $\widehat{SO(N)}$ and $\widehat{U(N)}$ *Kač-Moody* (KM) symmetries, respectively, were discovered before by Knizhnik [7] and Bershadsky [8]. It has been known for some time that the *unitary* series of Bershadsky-Knizhnik QSCAs does not admit nilpotent quantum BRST charges for any $N > 2$ [9], so that we are going to concentrate on the orthogonal series of QSCAs having the $SO(N)$ internal symmetry.³

2. The current contents of the 2d Bershadsky-Knizhnik orthogonal QSCA [7, 8] is given by the holomorphic fields $T(z)$, $G^i(z)$ and $J^a(z)$, all having the standard mode expansions

$$\begin{aligned} T(z) &= \sum_n L_n z^{-n-2} , \\ G^i(z) &= \sum_r G_r^i z^{-r-3/2} , \\ J^a(z) &= \sum_n J_n^a z^{-n-1} , \end{aligned} \tag{1}$$

and (conformal) dimensions 2, 3/2 and 1, respectively. The supercurrents G^i , $i = 1, \dots, N$, are defined in the fundamental representation of the internal symmetry group $SO(N)$ generated by the zero modes J_0^a , $a = 1, \dots, \frac{1}{2}N(N-1)$, in the adjoint

³A construction of quantum BRST charges for the orthogonal series of QSCAs was also briefly discussed in ref. [9], but the results presented there are, however, incomplete.

representation.

Most of the *operator product expansions* (OPEs) defining an orthogonal Bershadsky-Knizhnik QSCA take the standard linear form, *viz.*

$$\begin{aligned}
T(z)T(w) &\sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} , \\
T(z)G^i(w) &\sim \frac{\frac{3}{2}G^i(w)}{(z-w)^2} + \frac{\partial G^i(w)}{z-w} , \\
T(z)J^a(w) &\sim \frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w} , \\
J^a(z)G^i(w) &\sim \frac{(t^a)^{ij}G^j(w)}{z-w} , \\
J^a(z)J^b(w) &\sim \frac{f^{abc}J^c(w)}{z-w} + \frac{-k\delta^{ab}}{(z-w)^2} ,
\end{aligned} \tag{2}$$

where k is an arbitrary ‘level’ of the KM subalgebra, f^{abc} are $SO(N)$ structure constants, and $(t^a)^{ij}$ are generators of $SO(N)$ in the fundamental (vector) representation,

$$\begin{aligned}
[t^a, t^b] &= f^{abc}t^c, \quad f^{abc}f^{abd} = 2(N-2)\delta^{cd} , \\
\text{tr}(t^a t^b) &= -2\delta^{ab} , \quad (t^a)^{ij}(t^a)^{kl} = \delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk} .
\end{aligned} \tag{3}$$

On symmetry and dimensional reasons, the only (non-linear) OPE defining the supersymmetry piece of QSCA can be of the form

$$\begin{aligned}
G^i(z)G^j(w) &\sim a_1 \frac{\delta^{ij}}{(z-w)^3} + a_2 \frac{(t^a)^{ij}J^a(w)}{(z-w)^2} + \frac{1}{z-w} \left[2\delta^{ij}T(w) + \frac{1}{2}a_2(t^a)^{ij}\partial J^a(w) \right] \\
&+ \frac{1}{z-w} \left[a_3 \left(t^{(a} t^{b)} \right)^{ij} + a_4 \delta^{ab} \delta^{ij} \right] : J^a J^b : (w) ,
\end{aligned} \tag{4}$$

where a_1 , a_2 , a_3 and a_4 are parameters to be determined by solving the Jacobi identity, and the normal ordering is defined by

$$: J^a J^b : (w) = \lim_{z \rightarrow w} \left[J^a(z)J^b(w) + \frac{k\delta^{ab}}{(z-w)^2} \right] . \tag{5}$$

Indices in brackets mean symmetrization with unit weight, e.g. $t^{(a} t^{b)} \equiv \frac{1}{2}(t^a t^b + t^b t^a)$. Eq. (4) can be considered as the general *ansatz* for supersymmetry algebra.

Demanding consistency of the whole algebra determines the parameters [7, 8]:

$$a_1 = \frac{k(N+2k-4)}{N+k-3} , \quad a_2 = \frac{N+2k-4}{N+k-3} , \quad a_3 = a_4 = \frac{1}{N+k-3} , \tag{6}$$

while the Virasoro central charge of this QSCA is also quantized as [7, 8]:

$$c = \frac{k(N^2 + 6k - 10)}{2(N + k - 3)} . \quad (7)$$

The KM parameter k remains arbitrary in this construction.

In case of the $N = 2$ QSCA, the non-linearity actually disappears and the algebra becomes the $N = 2$ linear SCA, since the total coefficient in front of the sum of two last terms in the second line of eq. (4) vanishes ($i^2 + 1 = 0$) after substituting $U(1) \cong SO(2)$ and the last eq. (6). Therefore, the non-linear structure of Bershadsky-Knizhnik QSCAs only appears for $N \geq 3$.

3. Despite of the apparent non-linearity of QSCAs, their quantum BRST charges should be in correspondence with their classical BRST charges, up to renormalisation. The classical procedure is known for an arbitrary algebra of first-class constraints [10]. It was already used to obtain the quantum BRST charge for the non-linear W_3 algebra [11], and later generalised to any quadratically non-linear W -type algebra in ref. [9].

Consider a set of bosonic generators B_i and fermionic generators F_α , which satisfy a graded non-linear algebra of the form

$$\begin{aligned} \{B_i, B_j\}_{\text{P.B.}} &= f_{ij}{}^k B_k , \\ \{B_i, F_\alpha\}_{\text{P.B.}} &= f_{i\alpha}{}^\beta F_\beta , \\ \{F_\alpha, F_\beta\}_{\text{P.B.}} &= f_{\alpha\beta}{}^i B_i + \Lambda_{\alpha\beta}{}^{ij} B_i B_j , \end{aligned} \quad (8)$$

in terms of the graded Poisson (or Dirac) brackets, with some 3-point and 4-point ‘structure constants’, $f_{ij}{}^k$, $f_{i\alpha}{}^\beta$, $f_{\alpha\beta}{}^i$ and $\Lambda_{\alpha\beta}{}^{ij}$, respectively, which have to be ordinary numbers. The symmetry properties of these constants with respect to exchanging their indices obviously follow from their definition by eq. (8), and they are assumed below. When using the unified index notation, $A \equiv (i, \alpha), \dots$, the Jacobi identities for the classical graded algebra of eq. (8) take the form

$$\begin{aligned} f_{[AB}{}^D f_{C]D}{}^E &= 0 , \\ \Lambda_{[AB}{}^{DE} f_{C]D}{}^F + \Lambda_{[AB}{}^{DF} f_{C]D}{}^E + f_{[AB}{}^D \Lambda_{C]D}{}^{EF} &= 0 . \end{aligned} \quad (9)$$

As is clear from eq. (9), $f_{AB}{}^C$ are to be the structure constants of a graded Lie algebra.⁴ According to the classical BRST procedure,⁵ one introduces an anticommuting

⁴We assume that all symmetry operations with unified indices also have to be understood in the graded sense. In particular, a graded ‘antisymmetrisation’ of indices with unit weight (denoted by mixed brackets $[\]$ here) actually means the antisymmetrisation for bosonic-bosonic or bosonic-fermionic index pairs, but the symmetrisation for indices which are both fermionic.

⁵See, e.g., ref. [12] for a review.

ghost-antighost pair (c^m, b_m) for each of the bosonic generators B_m , and the commuting ghost-antighost pair $(\gamma^\alpha, \beta_\alpha)$ for each of the fermionic generators F_α . The ghosts satisfy (graded) bracket relations

$$\begin{aligned} \{c^m, c^n\}_{\text{P.B.}} &= \{b_m, b_n\}_{\text{P.B.}} = 0, \quad \{c^m, b_n\}_{\text{P.B.}} = \delta^m_n, \\ \{\gamma^\alpha, \gamma^\beta\}_{\text{P.B.}} &= \{\beta_\alpha, \beta_\beta\}_{\text{P.B.}} = 0, \quad \{\gamma^\alpha, \beta_\beta\}_{\text{P.B.}} = \delta^\alpha_\beta. \end{aligned} \quad (10)$$

Additional ghosts for the composite generators $B^i B^j$ are not needed since invariance of the classical theory under B^i already implies invariance under $B^i B^j$ [10].

The classical BRST charge Q is given by [10]

$$\begin{aligned} Q = & c^n B_n + \gamma^\alpha F_\alpha + \frac{1}{2} f_{ij}{}^k b_k c^j c^i + f_{i\alpha}{}^\beta \beta_\beta \gamma^\alpha c^i - \frac{1}{2} f_{\alpha\beta}{}^n b_n \gamma^\beta \gamma^\alpha \\ & - \frac{1}{2} \Lambda_{\alpha\beta}{}^{ij} B_i b_j \gamma^\alpha \gamma^\beta - \frac{1}{24} \Lambda_{\alpha\beta}{}^{ij} \Lambda_{\gamma\delta}{}^{kl} f_{ik}{}^m b_j b_l b_m \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta. \end{aligned} \quad (11)$$

Compared to the standard expression for the linear algebras ($\Lambda = 0$), the BRST charge of eq. (11) has the additional 3-(anti)ghost terms, dependent on the initial bosonic generators B_i , and the 7-(anti)ghost terms as well. It is easy to check that the classical ‘master equation’

$$\{Q, Q\}_{\text{P.B.}} = 0 \quad (12)$$

follows from eq. (9) and the related identity

$$\Lambda_{\{\alpha\beta}{}^{ij} \Lambda_{\gamma\delta}{}^{kl} f_{ik}{}^m = \Lambda_{\{\alpha\beta}{}^{ij} \Lambda_{\gamma\delta}{}^{[k} f_{ik}{}^{l]m}. \quad (13)$$

The classical BRST charge (11) may serve as the starting point in a construction of quantum BRST charge Q_{BRST} for the corresponding graded non-linear quantum algebra. Since we are actually interested in quantum QSCAs, we can assume that all operators are just currents, with a holomorphic dependence on z or, equivalently, with an additional affine index (see eq. (1), for example). In particular, in eq. (10) one should replace the (graded) Poisson brackets by (anti)commutators. In addition, in quantum theory, one must take into account central extensions and the normal ordering needed for defining products of bosonic generators. This results in the quantum (anti)commutation relations

$$\begin{aligned} [B_i, B_j] &= f_{ij}{}^k B_k + h_{ij} Z, \\ [B_i, F_\alpha] &= f_{i\alpha}{}^\beta F_\beta, \\ \{F_\alpha, F_\beta\} &= h_{\alpha\beta} Z + f_{\alpha\beta}{}^i B_i + \Lambda_{\alpha\beta}{}^{ij} : B_i B_j : , \end{aligned} \quad (14)$$

where the central charge generator Z commutes with all the other generators, and the new constants h_{ij} and $h_{\alpha\beta}$ are supposed to be restricted by the Jacobi identities. Although no general procedure seems to exist, which would explain how to ‘renormalise’

the naively quantised normally-ordered charge Q to a nilpotent quantum-mechanical operator Q_{BRST} , the answer is known for a particular class of quantum algebras of the W -type [9]. Similarly to the quantum W_3 algebra case considered in ref. [11], a non-trivial modification of eq. (11) in quantum theory essentially amounts to a *multiplicative* renormalisation of the structure constants $f_{\alpha\beta}{}^i$, namely

$$Q_{\text{BRST}} = c^n B_n + \gamma^\alpha F_\alpha + \frac{1}{2} f_{ij}{}^k : b_k c^j c^i : + f_{i\alpha}{}^\beta : \beta_\beta \gamma^\alpha : c^i - \frac{1}{2} \eta f_{\alpha\beta}{}^n b_n \gamma^\beta \gamma^\alpha \\ - \frac{1}{2} \Lambda_{\alpha\beta}{}^{ij} B_i b_j \gamma^\alpha \gamma^\beta - \frac{1}{24} \Lambda_{\alpha\beta}{}^{ij} \Lambda_{\gamma\delta}{}^{kl} f_{ik}{}^m b_j b_l b_m \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta . \quad (15)$$

This *ansatz* for the quantum BRST operator introduces only one additional renormalisation parameter η to be determined from the BRST charge nilpotency condition. Since the central extension parameters of the quantum non-linear algebra are severely restricted by the Jacobi identities, whereas the quantum BRST charge nilpotency condition could lead to some more restrictions on their values, this construction procedure could make them overdetermined, in general. Therefore, the existence of a quantum BRST charge is not guaranteed, and it is important to check consistency in each particular case.

4. For the BRST quantisation of Bershadsky-Knizhnik $SO(N)$ -based QSCA the following ghosts are needed:

- the conformal ghosts (b, c) , an anticommuting pair of world-sheet free fermions of conformal dimensions $(2, -1)$, respectively;
- the N -extended superconformal ghosts (β^i, γ^i) of conformal dimensions $(\frac{3}{2}, -\frac{1}{2})$, respectively, in the fundamental (vector) representation of $SO(N)$;
- the $SO(N)$ internal symmetry ghosts $(\tilde{b}^a, \tilde{c}^a)$ of conformal dimensions $(1, 0)$, respectively, in the adjoint representation of $SO(N)$.

The reparametrisation ghosts

$$b(z) = \sum_{n \in \mathbf{Z}} b_n z^{-n-2} , \quad c(z) = \sum_{n \in \mathbf{Z}} c_n z^{-n+1} , \quad (16)$$

have the following OPE and anticommutation relations:

$$b(z) c(w) \sim \frac{1}{z-w} , \quad \{c_m, b_n\} = \delta_{m+n,0} . \quad (17)$$

The superconformal ghosts

$$\beta^i(z) = \sum_{r \in \mathbf{Z} + 1/2} \beta_r^i z^{-r-3/2} , \quad \gamma^i(z) = \sum_{r \in \mathbf{Z} + 1/2} \gamma_r^i z^{-r+1/2} , \quad (18)$$

satisfy

$$\beta^i(z) \gamma^j(w) \sim \frac{-\delta^{ij}}{z-w}, \quad [\gamma_r^i, \beta_s^j] = \delta_{r+s,0}. \quad (19)$$

An integer or half-integer moding of these generators corresponds to the usual distinction between the Ramond- and Neveu-Schwarz-type sectors.

Finally, the fermionic $SO(N)$ ghosts

$$\tilde{b}^a(z) = \sum_{n \in \mathbf{Z}} \tilde{b}_n^a z^{-n-1}, \quad \tilde{c}^a(z) = \sum_{n \in \mathbf{Z}} \tilde{c}_n^a z^{-n}, \quad (20)$$

have

$$\tilde{b}^a(z) \tilde{c}^a(w) \sim \frac{\delta^{ab}}{z-w}, \quad \{\tilde{c}_m^a, \tilde{b}_n^b\} = \delta^{ab} \delta_{m+n,0}. \quad (21)$$

Eq. (15) provides us with the reasonable ansatz for the *quantum* BRST charge,⁶

$$\begin{aligned} Q_{\text{BRST}} = & c_{-n} L_n + \gamma_{-r}^i G_r^i + \tilde{c}_{-n}^a J_n^a - \frac{1}{2}(m-n) c_{-m} c_{-n} b_{m+n} + n c_{-m} \tilde{c}_{-n}^a \tilde{b}_{m+n}^a \\ & + \left(\frac{m}{2} - r\right) c_{-m} \beta_{m+r}^i \gamma_{-r}^i - b_{r+s} \gamma_{-r}^i \gamma_{-s}^i - \tilde{c}_{-m}^a \beta_{m+r}^i (t^a)^{ij} \gamma_{-r}^j \\ & + \eta a_2 (r-s) \tilde{b}_{r+s}^a \gamma_{-r}^i (t^a)^{ij} \gamma_{-s}^j - \frac{1}{2} f^{abc} \tilde{c}_{-m}^a \tilde{c}_{-n}^b \tilde{b}_{m+n}^c \\ & - \frac{1}{2} a_4 \left[(t^{(a} t^{b)})^{ij} + \delta^{ab} \delta^{ij} \right] J_{r+s+m}^a \tilde{b}_{-m}^b \gamma_{-r}^i \gamma_{-s}^j - \frac{1}{24} a_4^2 \left[(t^{(a} t^{b)})^{ij} + \delta^{ab} \delta^{ij} \right] \\ & \times \left[(t^{(c} t^{d)})^{kl} + \delta^{cd} \delta^{kl} \right] f^{ace} \delta_{m+n+p, r+s+t+u} \tilde{b}_m^b \tilde{b}_n^d \tilde{b}_p^e \gamma_{-r}^i \gamma_{-s}^j \gamma_{-t}^k \gamma_{-u}^l, \end{aligned} \quad (22)$$

where a quantum renormalisation parameter η has been introduced. Its value is going to be fixed by the BRST charge nilpotency conditions. The coefficients a_2 and a_4 have already been fixed by eq. (6).

We find always useful to represent a quantum BRST charge as

$$Q_{\text{BRST}} = \oint_0 \frac{dz}{2\pi i} j_{\text{BRST}}(z), \quad (23)$$

where the BRST current $j_{\text{BRST}}(z)$ is defined *modulo* total derivative.⁷ In particular, the BRST current $j_{\text{BRST}}(z)$ corresponding to the BRST charge of eq. (22) is given by

$$\begin{aligned} j_{\text{BRST}}(z) = & cT + \gamma^i G^i + \tilde{c}^a J^a + bc\partial c - \tilde{c}^a \partial \tilde{c}^a - \frac{1}{2} c \gamma^i \partial \beta^i - \frac{3}{2} c \beta^i \partial \gamma^i - b \gamma^i \gamma^i \\ & - \eta a_2 \tilde{b}^a (t^a)^{ij} (\gamma^i \partial \gamma^j - \gamma^j \partial \gamma^i) - \tilde{c}^a (t^a)^{ij} \beta^i \gamma^j - \frac{1}{2} f^{abc} \tilde{c}^a \tilde{c}^b \tilde{b}^c \\ & - \frac{1}{2} a_4 \left[(t^{(a} t^{b)})^{ij} + \delta^{ab} \delta^{ij} \right] J^a \tilde{b}^b \gamma^i \gamma^j - \frac{1}{24} a_4^2 \left[(t^{(a} t^{b)})^{ij} + \delta^{ab} \delta^{ij} \right] \\ & \times \left[(t^{(c} t^{d)})^{kl} + \delta^{cd} \delta^{kl} \right] f^{ace} \tilde{b}^b \tilde{b}^d \tilde{b}^e \gamma^i \gamma^j \gamma^k \gamma^l. \end{aligned} \quad (24)$$

⁶The normal ordering is implicit.

⁷The total derivative can be fixed by requiring the $j_{\text{BRST}}(z)$ to transform as a primary field.

The most tedious part of calculational handwork in computing Q_{BRST}^2 can be avoided when using either the Mathematica Package for computing OPEs [13] or some of the general results in ref. [9]. In particular, as was shown in ref. [9], quantum renormalisation of the 3-point structure constants in the quantum BRST charge should be *multiplicative*, whereas the non-linearity 4-point ‘structure constants’ should *not* be renormalised at all — the facts already used in the BRST charge ansatz above. Most importantly, among the contributions to the Q_{BRST}^2 , only the terms *quadratic* in the ghosts are relevant. Their vanishing imposes the constraints on the central extension coefficients of the QSCA and simultaneously determines the renormalization parameter η . The details can be found in the appendices of ref. [9]. The same conclusion comes as a result of straightforward calculation on computer. Therefore, finding out the nilpotency conditions amounts to calculating only a few terms ‘by hands’, namely, those which are quadratic in the ghosts. This makes the whole calculation as simple as that in ordinary string theories based on linear SCAs [5].

The 2-ghost terms in the Q_{BRST}^2 arise from single contractions of the first three linear (in the ghosts) terms of Q_{BRST} with themselves and with the next cubic terms of eq. (24), and from double contractions of the latter among themselves. They result in the pole contributions to $j_{\text{BRST}}(z)j_{\text{BRST}}(w)$, proportional to $(z - w)^{-n}$ with $n = 1, 2, 3, 4$. All the residues have to vanish modulo total derivative. We find

$$\begin{aligned}
j_{\text{BRST}}(z)j_{\text{BRST}}(w) \sim & \frac{c(z)c(w)}{2(z-w)^4} \left[c - N^2 + 12N - 26 \right] \\
& + \frac{\gamma^i(z)\gamma^i(w)}{(z-w)^3} \left[a_1 - \frac{ka_4}{2}(N-1)(N-2) - 4\eta a_2(N-1) + 2 \right] \\
& + \frac{\tilde{c}^a(z)\tilde{c}^a(w)}{(z-w)^2} [-k - 2(N-2) + 2] \\
& + \frac{J^a(w)(t^a)^{ij}\gamma^i(w)\partial\gamma^j(w)}{z-w} \left[-4\eta a_2 - 4a_4 \left(1 - \frac{N}{2} \right) \right] + \dots,
\end{aligned} \tag{25}$$

where the dots stand for the other terms of higher order in (anti)ghosts, and the coefficients a_1 , a_2 , a_4 and c are given by eqs. (6) and (7), respectively. Eq. (25)

immediately yields the BRST charge nilpotency conditions:

$$\begin{aligned}
c_{\text{tot}} &\equiv c + c_{\text{gh}} = \frac{k(N^2 + 6k - 10)}{2(N + k - 3)} - N^2 + 12N - 26 = 0 , \\
s_{\text{tot}} &\equiv a_1 + (a_1)_{\text{gh}} = \frac{k(N + 2k - 4)}{N + k - 3} - \frac{k(N - 1)(N - 2)}{2(N + k - 3)} \\
&\quad - \frac{4\eta(N - 1)}{N + k - 3} + 2 = 0 , \\
k_{\text{tot}} &\equiv k + k_{\text{gh}} = k + 2N - 6 = 0 , \\
&\quad \frac{\eta(N + 2k - 4)}{N + k - 3} - \frac{N - 2}{2(N + k - 3)} = 0 .
\end{aligned} \tag{26}$$

The first line of eq. (26) just means the vanishing total central charge, where the value of c_{gh} is dictated by the standard formula of conformal field theory [5]

$$\begin{aligned}
c_{\text{gh}} &= 2 \sum_{\lambda} n_{\lambda} (-1)^{2\lambda+1} (6\lambda^2 - 6\lambda + 1) \\
&= 1 \times (-26) + N \times (+11) + \frac{1}{2}N(N - 1) \times (-2) = -26 + 12N - N^2 ,
\end{aligned} \tag{27}$$

λ is conformal dimension and n_{λ} is a number of the conjugated ghost pairs: $\lambda = 2, 3/2, 1$ and $n_{\lambda} = 1, N, \frac{1}{2}N(N - 1)$, respectively. The zero central charge condition alone has two solutions,

$$k = 6 - 2N , \quad \text{and} \quad 6k = N^2 - 12N + 26 , \tag{28}$$

but only the first of them is compatible with the third equation (26).

Central extensions (anomalies) of the ghost-extended QSCA need not form a linear supermultiplet, and they actually do not. Therefore, the vanishing central charge alone does not imply the other equations (26) to be automatically satisfied, unlike in the linear case. The last equation (26) just determines the renormalisation parameter η . Finally, the second equation (26) can be interpreted as the vanishing total supersymmetric anomaly. Since the supersymmetry is non-linearly realised, this anomaly does not have to vanish as a consequence of the other equations (26), but restricts N as the only remaining parameter. Substituting

$$k = 6 - 2N , \quad \text{and} \quad \eta = \frac{N - 2}{2(8 - 3N)} , \tag{29}$$

into the second equation (26), we find

$$6(N - 3) + \frac{(N - 1)(N - 2)(N - 5)}{N - 3} = 0 , \tag{30}$$

which has only one solution, $N = 4$. Therefore, though the system of four equations (26) for only three parameters η , k and N is clearly overdetermined (while N is a positive integer!), there is still the only solution, namely

$$N = 4 \ , \quad k = -2 \ , \quad \eta = -\frac{1}{4} \ . \quad (31)$$

5. In our letter we constructed the quantum BRST charges for the orthogonal series of Bershadsky-Knizhnik non-linear QSCAs. The BRST charge nilpotency conditions cannot be met, unless $N = 4$ and $k = -2$. This is apparently in line with the analogous fact [9] that the BRST quantisation breaks down for all unitary $U(N)$ -based Bershadsky-Knizhnik QSCAs of $N \geq 3$, since their BRST charge nilpotency conditions are always in conflict with the Jacobi identities.⁸

The existence of the nilpotent quantum BRST charge for the non-linear $SO(4)$ -based Bershadsky-Knizhnik QSCA implies the existence of a new W -type string theory with the non-linearly realised $N = 4$ world-sheet supersymmetry. In quantum theory, the vanishing currents can be interpreted as operator constraints on physical states. By interpreting zero modes of scalar fields in the matter QSCA realisations as spacetime coordinates, one arrives at a first-quantised description of string oscillations. Unfortunately, gauging the local symmetries of the $SO(4)$ -based Bershadsky-Knizhnik QSCA results in the positive total ghost central charge contribution, $c_{\text{gh}} = 6$. In addition, the anomaly-free solution requires $k = -2 < 0$. Therefore, there seems to be no way to build an anomaly-free string theory when using only unitary representations of the $SO(4)$ -based Bershadsky-Knizhnik QSCA. When choosing a non-unitary representation of this QSCA with $k = -2$, one can get the desired anomaly-free matter contribution, $c_{\text{m}} = -6$, but then a space-time interpretation together with a physical significance of the construction, if any, become obscure. Despite of all this, we believe that it is worthy to know how many string models, consistent from the mathematical point of view, can be constructed.

References

- [1] M. Ademollo et al., Phys. Lett. **62B** (1976) 105; Nucl. Phys. **B111** (1976) 77.
- [2] M. Pernici and P. van Nieuwenhuizen, Phys. Lett. **169B** (1986) 381.

⁸The $U(N)$ -based Bershadsky-Knizhnik QSCAs do not admit unitary representations for $N \geq 3$ also [14].

- [3] N. Berkovits and C. Vafa, Mod. Phys. Lett. **A9** (1994) 653.
- [4] S. V. Ketov, Class. and Quantum Grav. **10** (1993) 1689.
- [5] S. V. Ketov, *Conformal Field Theory*, Singapore: World Scientific, 1994.
- [6] E. S. Fradkin and V. Ya. Linetsky, Phys. Lett. **282B** (1992) 352.
- [7] V. G. Knizhnik, Theor. Math. Phys. **66** (1986) 68.
- [8] M. Bershadsky, Phys. Lett. **174B** (1986) 285.
- [9] K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, Commun. Math. Phys. **124** (1989) 87.
- [10] E. S. Fradkin and T. E. Fradkina, Phys. Lett. **72B** (1978) 343.
- [11] J. Thierry-Mieg, Phys. Lett. **197B** (1987) 368.
- [12] M. Henneaux, Phys. Rep. **126** (1985) 1.
- [13] K. Thielemans, *A MathematicaTM Package for Computing Operator Product Expansions (OPEdefs 3.0)*, Leuven preprint, July 1992; Int. J. Mod. Phys. **C2** (1991) 787.
- [14] K. Schoutens, Nucl. Phys. **B292** (1987) 150.